Bounds on the number of edges in hypertrees

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Let $\mathcal{H}$ be a k-uniform hypergraph. A chain in $\mathcal{H}$ is a sequence of its vertices such that every k consecutive vertices form an edge. In 1999 Gyula Y. Katona and Hal Kierstead suggested to use chains in hypergraphs as the generalisation of paths [4]. Although, a number of results have been published on Hamilton-chains in recent years, the generalisation of trees with chains has remained an open area.

We generalise the concept of trees for uniform hypergraphs [5]. We say that a k-uniform hypergraph $\mathcal{F}$ is a hypertree if every two vertices of $\mathcal{F}$ are connected with a chain and an appropriate kind of cycle-free property holds. An edge-minimal/maximal hypertree is a hypertree whose edge set is minimal/maximal with respect to inclusion.

After considering these definitions, we show that a k-uniform hypertree on n vertices has at least $n-(k-1)$ edges up to a finite number of exceptions, and it has at most $\binom{n}{k-1}$ edges. We prove the asymptotic sharpness of these bounds.

A hypertree is called an l-hypertree if every chain within has length at most l. We investigate the maximal edge-number of k-uniform l-hypertrees, particularly in the case of $l=2$. We give a general construction with its consequences [7].

We give an upper bound on the edge-number of k-uniform edge-minimal hypertrees and conjecture that $\frac{1}{k-1}\binom{n}{2}$ is an upper bound. We show that if the conjecture is true, then it is sharp in asymptotic sense.

We use de Caen's Turn-type theorem for hypergraphs [2] to show that a k-uniform edge-maximal hypertree has at least $\frac{1}{k(k-1)}\frac{n-k+1}{n-k+2}\binom{n}{k-1}$ edges. After considering a construction in 3-uniform case, we formulate a conjecture about the asymptotic edge-number of edge-maximal hypertrees.

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