Hypergraph extensions of the Erdős-Gallai Theorem

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(joint work with Ervin Győri and Nathan Lemons)

Our goal is to extend the following result of Erdős and Gallai for hypergraphs:

Theorem 1 (Erdős-Gallai [1]) *Let G be a graph on n vertices containing no path of length k. Then $e(G) \leq \frac{1}{2}(k-1)n$. Equality holds iff G is the disjoint union of complete graphs on k vertices.*

We consider several generalizations of this theorem for hypergraphs. This is due to the fact that there are several possible ways to define paths in hypergraphs. One such definition of paths in hypergraphs is due to Berge.

Definition 2 *A Berge path of length k in a hypergraph is a collection of k hyperedges $h_1, \ldots, h_k$ and $k+1$ vertices $v_1, \ldots, v_{k+1}$ such that for each $1 \leq i \leq k$ we have $v_i, v_{i+1} \in h_i$.*

We find the extremal sizes of r-uniform hypergraphs avoiding Berge cycles of length k. Interestingly, the size of the extremal hypergraphs depend on the relationship between r and k. Specifically, we distinguish between the cases when $k \leq r$ and when $k > r$.

Theorem 3 *Fix $r > 2$, let $k > r$ and let $\mathcal{H}$ be a hypergraph containing no Berge path of length k. Then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$.*

On the other hand, if $k \leq r$, we have the following theorem.

Theorem 4 *Fix $2 < k \leq r$. If $\mathcal{H}$ is an r-uniform hypergraph with no path of length k, then $e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}$.*

We also present similar results for t-tight and tigth paths.

References

[1] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.