Chip games and paintability of complete multipartite graphs

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Paintability is a graph parameter (introduced by Schauz and Zhu in [2, 3]) which brings the classical notion of choosability (list colorability) to the on-line setting. How much the choosability and the paintability are related is an open problem. The paintability is never smaller, but there is no known example in which those two parameters differ by more than one. The efforts to show a bigger separation are currently focused on the study of complete multipartite graphs.

The paintability of a complete bipartite graph $K_{N,N}$ can be defined using the following simple chip game between Maker and Remover. We consider a board consisting of two directed paths of length $n + 1$. We place N chips on each of the two starting vertices. Then the game is played in rounds. In each round Maker chooses an arbitrary subset of the chips and moves them one step forward on the paths. Remover chooses one of the paths and removes from this path all the chips moved by Maker in this round. Maker wins the game if a chip reaches the end vertex of any path. Remover wins when there are no more chips on the board and Maker has not won. $K_{N,N}$ is n-paintable if and only if Remover has a winning strategy in the presented chip game.

An equivalent chip game has been considered earlier (by Aslam and Dhagat in [1]) as a model for on-line 2-coloring of uniform hypergraphs. In particular, the researchers were interested in the case of hypergraphs with bounded vertex degree, which translates to bounding the number of chips Maker is allowed to move in each round.

There is a simple winning strategy for Remover when $N = O(2^n)$. We show a winning strategy for Maker when $N = \Omega(n2^n)$. In the restricted variant in which Maker is allowed to move at most two chips in each round, we present winning strategies for both players – for Remover when $N = O(\phi^{2n})$, and for Maker when $N = \Omega(\phi^{2n})$ (ϕ is the golden ratio).

References

[1] J.A. Aslam, A. Dhagat, On-line algorithms for 2-coloring hypergraphs via chip games, Theor. Comput. Sci. 112 (1993), 355–369.
[2] U. Schauz, Mr. Paint and Mrs. Correct, Electron. J. Combin. 16:1 (2009), R77.
[3] X. Zhu, On-line list colouring of graphs, Electron. J. Combin. 16:1 (2009), R127.