Variations on the theorem of Thue

Jarek Grytczuk

A *repetition* is a sequence consisting of two identical blocks, like 123123. A *segment* in a sequence S is a subsequence of consecutive terms of S. A sequence S is *nonrepetitive* if none of its segments is a repetition. The theorem of Thue from 1906 asserts that there exist arbitrarily long nonrepetitive sequences over just three symbols. I will present some old and new problems inspired by this result. Below is a selection of just three of them.

Two identical subsequences on disjoint sets of indices occupying a segment in S are called a pair of *tight twins*. For instance, 121323 is a pair of tight twins which is not a repetition. How many symbols are needed to built arbitrarily long sequences avoiding tight twins?

Here is another one in the spirit of Erdős. Suppose we are given a collection of necklaces (sequences) $C = \{S_1, S_2, \ldots, S_n\}$ consisting of beads in several colors. A *fair division* of C is a partition $C = C_1 \cup C_2$ such that each part C_i captures the same number of beads of every color. The *splitting number* of a sequence S is the least number of cuts needed to obtain a collection of segments of S that have a fair division (if there is no such cutting we agree that the splitting number is infinite). For instance, the splitting number of 123132 is equal to one, as cutting the sequence in the middle does the job. Now, given a number k, how many colors are needed to construct an infinite sequence with no segments of splitting number at most k?

And another one in the spirit of Rota. A sequence of real numbers consisting of n blocks of length n each, is called *singular* if the matrix formed out of these blocks is singular, like 123213112. Is there an infinite sequence on some finite set of symbols (numbers) with no singular segment?