Nowhere-zero flows on signed regular graphs

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(joint work with Michael Schubert)

A signed graph (G, σ) is a graph G together with a function $\sigma : E(G) \to \{\pm 1\}$, which is called a signature of G. The set $N_\sigma = \{e : \sigma(e) = -1\}$ is the set of negative edges of (G, σ) and $E(G) - N_\sigma$ the set of positive edges. We study flows on signed graphs, and $F_c((G, \sigma))$ $(F((G, \sigma)))$ denotes the circular (integer) flow number of (G, σ).

Bouchet [1] conjectured that $F((G, \sigma)) \leq 6$ for every flow-admissible signed graph. This conjecture is equivalent to its restriction on cubic graphs. We prove this conjecture for flow-admissible cubic graphs that have three 1-factors such that any two of them induce a hamiltonian circuit of G. In particular, every flow-admissible uniquely 3-edge-colorable cubic graph has a nowhere-zero 6-flow.

For a graph G and $X \subseteq E(G)$ let $\Sigma_X(G)$ be the set of signatures σ of G, for which (G, σ) is flow-admissible and $N_\sigma \subseteq X$. Define $\mathcal{S}_X(G) = \{r :$ there is a signature $\sigma \in \Sigma_X(G)$ such that $F_c((G, \sigma)) = r\}$ to be the X-flow spectrum of G. The $E(G)$-flow spectrum is the flow spectrum of G and it is denoted by $\mathcal{S}(G)$. If we restrict our studies on integer-valued flows, then $\overline{\mathcal{S}}_X(G)$ denotes the integer X-flow spectrum of G.

We study the integer flow spectrum of signed cubic graphs G. There are cubic graphs whose integer flow spectrum does not contain 5 or 6. But we show, that $\{3, 4\} \subseteq \overline{\mathcal{S}}(G)$, for every bridgeless cubic graph $G \neq K_2^3$, where K_2^3 is the unique cubic graphs on two vertices. We construct an infinite family of bridgeless cubic graphs with integer flow spectrum $\{3, 4, 6\}$.

We further study the flow spectrum of $(2t + 1)$-regular graphs $(t \geq 1)$. In [2] it is proven that a $(2t+1)$-regular graph G is bipartite if and only if $F_c((G, \emptyset)) = 2+\frac{1}{t}$. Furthermore, if G is not bipartite, then $F_c((G, \emptyset)) \geq 2 + \frac{2}{2t-1}$. We extend this kind of result to signed $(2t + 1)$-regular graphs. Let $r \geq 2$ be a real number and G be a graph. A set $X \subseteq E(G)$ is r-minimal if

(1) there is a signature σ of G such that $F_c((G, \sigma)) = r$ and $N_\sigma = X$, and

(2) $F_c((G, \sigma')) \neq r$ for every signature σ' of G with $N_{\sigma'} \subset X$.

We show for $(2t + 1)$-regular graphs G, which have a t-factor: A set $X \subseteq E(G)$ is $(2 + \frac{1}{t})$-minimal if and only if X is a minimal set such that $G - X$ is bipartite.

Furthermore, if $X \subseteq E(G)$ is a $(2+\frac{1}{t})$-minimal set and $r \in \mathcal{S}_X(G)$, then $r = 2+\frac{1}{t}$ or $r \geq 2 + \frac{2}{2t-1}$.

References

[1] A. Bouchet, Nowhere-zero integral flows on bidirected graph, J. Comb. Theory Ser. B 34 (1983), 279–292.

[2] E. Steffen, Circular flow numbers of regular multigraphs, J. Graph Theory 36 (2001), 24–34.