Digraph partitions and full homomorphism dualities

César Hernández-Cruz

(joint work with Pavol Hell)

Let $D = (V, A)$ be a digraph. A *strong clique* of D is a set C of vertices such that for any two distinct vertices $x, y \in C$ both arcs $(x, y), (y, x)$ are in D. Let S, S' be two disjoint sets of vertices of D: we say that S is *completely adjacent to* S' (or S' is completely adjacent from S) if for any $x \in S, x' \in S'$, the pair (x, x') is an arc of D; we say that S is *completely non-adjacent to* S' (or S' is completely non-adjacent from S) if for any $x \in S, x' \in S'$, the pair (x, x') is not an arc of D. Let M be a fixed $\{0, 1\}$ matrix of size m, with k diagonal 0's and ℓ diagonal 1's. An *M-partition* of a digraph D is a partition of its vertex set $V(D)$ into parts $V_1, V_2, \ldots, V_{k+\ell}$ such that

- V_i is an independent set of D if $M(i, i) = 0$
- V_i is a strong clique of D if $M(i, i) = 1$
- V_i is completely non-adjacent to V_j if $M(i, j) = 0$
- V_i is completely adjacent to V_j if $M(i, j) = 1$

A *full homomorphism* of a digraph D to a digraph H is a mapping $f : V(D) \to V(H)$ such that for vertices $x \neq y$, $(x, y) \in A(D)$ if and only if $(f(x), f(y)) \in A(H)$. If H denote the digraph whose adjacency matrix is M, then D admits an M-partition if and only if it admits a full homomorphism to H.

Undirected graphs are viewed as special cases of digraphs, i.e., each edge xy is viewed as the two arcs $(x, y), (y, x)$. For a symmetric $\{0, 1\}$ matrix M, the same definition applies to define an M-partition of a graph G [3]. It is shown in [1, 2] that for any symmetric $\{0, 1\}$ matrix M there is a finite set $\mathcal{G}$ of graphs such that G admits an M-partition if and only if it does not contain an induced subgraph isomorphic to a member of $\mathcal{G}$. Alternately [3], we define a *minimal obstruction* to M-partition to be a digraph D which does not admit an M-partition, but such that for any vertex v of D, the digraph $D - v$ does admit an M-partition. Each symmetric $\{0, 1\}$ matrix M has only finitely many minimal graph obstructions [1, 2]. It was known these obstructions have at most $(k+1)(\ell+1)$ vertices [2] and this bound is best possible; however, the minimum upper bound has ben open for digraphs. We prove that in fact also each minimal digraph obstruction has at most $(k + 1)(\ell + 1)$ vertices (and this is best possible). We interpret our results as certain dualities of full homomorphisms, in the spirit of [1].

References

[1] R.N. Ball, J. Nešetřil, A. Pultr, Dualities in full homomorphisms, European J. Combin. 31 (2010), 106–119.

[2] T. Feder, P. Hell, On realizations of point determining graphs and obstructions to full homomorphisms, Discrete Math. 308 (2008), 1639–1652.

[3] P. Hell, Graph partitions with prescribed patterns, European J. Combin., to appear.